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Correctors for the Homogenization of Monotone Parabolic Operators

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Abstract

In the homogenization of monotone parabolic partial differential equations with oscillations in both the space and time variables the gradients converges only weakly in L^p. In the present paper we construct a family of correctors, such that, up to a remainder which converges to zero strongly in L^p, we obtain strong convergence of the gradients in L^p.

1 Introduction

In [8] the asymptotic behaviour (as $\epsilon \to 0$) of the solutions u_ϵ to a sequence of initial-boundary value problems of the form

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t} - \operatorname{div}(a(\dfrac{x}{\epsilon}, \dfrac{t}{\epsilon^\mu}, Du_\epsilon)) = f \text{ in } \Omega \times]0, T[, \\ u_\epsilon(x, 0) = u_0(x), \\ u_\epsilon(x, t) = 0 \text{ in } \partial\Omega \times]0, T[, \end{cases} \tag{1.1}$$

is studied. Here Ω is an open bounded set in $\mathbf{R}^{\mathrm{N}}$, T is a positive real number, $2 \le p < \infty$ and $\mu > 0$. Under the assumption that $a(\frac{x}{\epsilon}, \frac{t}{\epsilon^\mu}, Du_\epsilon)$ is $\epsilon-$ and ϵ^μ-periodic in the first and second variable, respectively, it is proved that

$$u_\epsilon \to u \text{ weakly in } \mathrm{L}^p(0, \mathrm{T}; \mathrm{W}_0^{1,p}(\Omega)),$$
$$a(\tfrac{x}{\epsilon}, \tfrac{t}{\epsilon^\mu}, Du_\epsilon) \to b(Du) \text{ weakly in } \mathrm{L}^{p'}(0, \mathrm{T}; \mathrm{L}^{p'}(\Omega; \mathbf{R}^{\mathrm{N}})),$$

where $1/p + 1/p' = 1$ and where u denotes the unique solution to

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(b(Du)) = f \text{ in } \Omega \times]0, T[, \\ u(x, 0) = u_0(x), \\ u(x, t) = 0 \text{ in } \partial\Omega \times]0, T[, \end{cases} \tag{1.2}$$

where the limit map b in (1.2) only depends on the sequence $(a(\cdot/\epsilon, \cdot/\epsilon^{\mu}, \xi))$ and on μ and where b is different for $0 < \mu < 2$, $\mu = 2$ and $\mu > 2$, respectively. This result implies that

$$Du_{\epsilon}(x,t) = Du(x,t) + r_{\epsilon}(x,t),$$

where the remainder r_{ϵ} converges to zero only weakly in $\mathrm{L}^p(0,\mathrm{T};\mathrm{L}^p(\Omega;\mathbf{R}^{\mathrm{N}}))$. The purpose of the present paper is to construct a family of correctors $(p_{\epsilon}) = (p_{\epsilon}(x,t,\xi))$ such that

$$p_{\epsilon}(\cdot,\cdot,\xi) \to \xi \text{ weakly in } \mathrm{L}^p(0,\mathrm{T};\mathrm{L}^p(\Omega;\mathbf{R}^{\mathrm{N}}))$$

for every $\xi \in \mathbf{R}^{\mathbf{N}}$ and

$$Du_{\epsilon}(x,t) = p_{\epsilon}(x,t,(M_{\epsilon}Du)(x,t)) + r_{\epsilon}(x,t),$$

where the remainder r_{ϵ} converges to zero strongly in $\mathrm{L}^p(0,\mathrm{T};\mathrm{L}^p(\Omega;\mathbf{R}^{\mathrm{N}}))$ and where (M_{ϵ}) is a sequence of linear operators on $\mathrm{L}^p(0,\mathrm{T};\mathrm{L}^p(\Omega;\mathbf{R}^{\mathrm{N}}))$ which converges strongly to the identity map on $\mathrm{L}^p(0,\mathrm{T};\mathrm{L}^p(\Omega;\mathbf{R}^{\mathrm{N}}))$. The results presented in this article are rather technical and involves numerous estimates on small ϵ-cubes. But the implications from Theorem 2.1 are important. In particular for computational modeling of (1.1) and (1.2) since it implies strong convergence of the gradients in the energy norm. In a simplified way we can say that the improvement of the convergence lies in the fact that the local behaviour on the ϵ-cubes are added to the homogenized solution. Heuristically this amounts to adding the second term in an asymptotic expansion, see [1]. The corrector problem was first studied in [1] for linear elliptic and parabolic problems. For a careful study of linear parabolic problems we refer to [3]. See also [2] and [6]. The extension to the monotone elliptic case is performed in [4]. The present work is very much inspired by the methods developed in [4]. The paper is organized as follows: Section 2 contains some preliminaries and in Section 3 we present the main theorem (Theorem 3.1). In Section 4 we collect some useful estimates for the correctors and in Section 5 we give the proof of Theorem 3.1.

2 Preliminaries

Throughout this paper we will denote by Ω a bounded open set in $\mathbf{R}^{\mathrm{N}}$ and we will let $\mathrm{V} = \mathrm{W}_0^{1,p}(\Omega)$, with norm $\|u\|_V^p = \int_{\Omega} |Du|^p \, dx$ and $\mathrm{V}' = \mathrm{W}^{-1,p'}(\Omega)$.

We consider the evolution triple

$$\mathrm{V} \subseteq \mathrm{L}^2(\Omega) \subseteq \mathrm{V}',$$

with dense embeddings. Further, for positive real-valued T and for $2 \le p < \infty$, we define $\mathcal{V} = \mathrm{L}^p(0,\mathrm{T};\mathrm{V})$ and $\mathcal{V}' = \mathrm{L}^{p'}(0,\mathrm{T};\mathrm{V}')$, where $1/p + 1/p' = 1$ and the corresponding evolution triple

$$\mathcal{V} \subseteq \mathrm{L}^2(]0,\mathrm{T}[\times\Omega) \subseteq \mathcal{V}'$$

also with dense embeddings where the duality pairing $\langle \cdot,\cdot \rangle_{\mathcal{V}}$ between $\mathcal{V}$ and $\mathcal{V}'$ is given by

$$\langle f,u \rangle_{\mathcal{V}} = \int_0^T \langle f(t),u(t) \rangle_V \, dt, \text{ for } u \in \mathcal{V},\ f \in \mathcal{V}'.$$

Given $u_0 \in \mathrm{L}^2(\Omega)$, the space $\mathcal{W}_0$ is defined as

$$\mathcal{W}_0 = \{v \in \mathcal{V} : v' \in \mathcal{V}' \text{ and } v(0) = u_0 \in \mathrm{L}^2(\Omega)\}.$$

Here v' denotes the time derivative of v, which is to be taken in distributional sense. Moreover, we define

$$\mathcal{U} = \mathrm{L}^p(0,\mathrm{T};\mathrm{L}^p(\Omega;\mathbf{R}^\mathrm{N})) \text{ and } \mathcal{U}' = \mathrm{L}^{p'}(0,\mathrm{T};\mathrm{L}^{p'}(\Omega;\mathbf{R}^\mathrm{N})).$$

with the duality pairing

$$\langle u,v\rangle_{\mathcal{U}} = \int_0^T \int_\Omega (u,v)\,dxdt, \quad \text{for } u \in \mathcal{U}' \text{ and } v \in \mathcal{U},$$

where $(\cdot,\cdot)$ denotes the scalar product in $\mathbf{R}^\mathrm{N}$. By $|\cdot|$ we understand the usual Euclidean norm in $\mathbf{R}^\mathrm{N}$ and by $m(\cdot)$ we understand the Lebesgue measure. Moreover, by (ϵ) we understand a sequence of positive real numbers tending to 0^+.

Let $Y =]0,1[^\mathrm{N}$ be the unit cube in $\mathbf{R}^\mathrm{N}$ and let $Y \times T_0 =]0,1[^\mathrm{N} \times]0,1[$ be the unit cube in $\mathbf{R}^\mathrm{N} \times \mathbf{R}_+$.

Definition 2.1. We say that a function $u : \mathbf{R}^\mathrm{N} \times]0,\mathrm{T}[\to \mathbf{R}$ is Y-*periodic* if $u(x+e_i,t) = u(x,t)$ for every $x \in \mathbf{R}^\mathrm{N}$, $t \in]0,\mathrm{T}[$ and for every $i = 1,\dots,\mathrm{N}$, where (e_i) is the canonical basis of $\mathbf{R}^\mathrm{N}$. Further, we say that a function $u : \mathbf{R}^\mathrm{N} \times \mathbf{R}_+ \to \mathbf{R}$ is $Y \times T_0$-*periodic* if $u(x+e_i,t) = u(x,t) = u(x,t+1)$ for every $x \in \mathbf{R}^\mathrm{N}$, $t \in \mathbf{R}_+$ and for every $i = 1,\dots,\mathrm{N}$.

We consider the following spaces of periodic functions:

$$\mathrm{V}_{\sharp,Y} = \{u \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbf{R}^\mathrm{N}) : u \text{ is Y-periodic and has mean value zero over Y}\},$$

and

$$\mathcal{V}_{\sharp,Y\times\ T_0} = \{u \in \mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}_+;\mathrm{V}_{\sharp,Y}) : u \text{ is } T_0-\textit{periodic}\}.$$

Definition 2.2. Given $0 < \alpha \leq 1$, $2 \leq p < \infty$ and three positive real constants c_0, c_1 and c_2 we define the class $S_{\sharp,Y\times T_0} = S_{\sharp,Y\times T_0}(c_0,c_1,c_2,\alpha)$ of maps

$$a : \mathbf{R}^\mathrm{N} \times \mathbf{R}_+ \times \mathbf{R}^\mathrm{N} \to \mathbf{R}^\mathrm{N},$$

such that

(i) $a(\cdot,\cdot,\xi)$ is $Y \times \tau_0$-periodic for every $\xi \in \mathbf{R}^\mathrm{N}$,

(ii) $|a(y,\tau,0)| \leq c_0$ a.e in $\mathbf{R}^\mathrm{N} \times \mathbf{R}_+$,

(iii) $a(\cdot,\cdot,\xi)$ is Lebesgue measurable for every $\xi \in \mathbf{R}^\mathrm{N}$,

(vi) $|(a(y,\tau,\xi_1) - a(y,\tau,\xi_2)| \leq c_1(1+|\xi_1|+|\xi_2|)^{p-1-\alpha}|\xi_1-\xi_2|^\alpha$, a.e. in $\mathbf{R}^\mathrm{N} \times \mathbf{R}_+$ for all $\xi_1,\xi_2 \in \mathbf{R}^\mathrm{N}$,

(v) $(a(y,\tau,\xi_1) - a(y,\tau,\xi_2), \xi_1-\xi_2) \geq c_2|\xi_1-\xi_2|^p$, a.e. in $\mathbf{R}^\mathrm{N} \times \mathbf{R}_+$ for all $\xi_1,\xi_2 \in \mathbf{R}^\mathrm{N}$, $\xi_1 \neq \xi_2$.

We recall some results for maps $a \in S_{\sharp,Y\times T_0}$:

Proposition 2.1. *Suppose that $a \in S_{\sharp,Y\times T_0}$. Then, for every $f \in \mathcal{V}'$ and for every $\epsilon > 0$, (1.1) possesses a unique solution $u_\epsilon \in \mathcal{W}_0 \cap L^\infty(0,T;L^2(\Omega))$.*

Proof. See e.g. [10].

Proposition 2.2. *Let us put $a_\epsilon(\cdot,\cdot,\xi) = a(\frac{\cdot}{\epsilon},\frac{\cdot}{\epsilon^\mu},\xi)$. Suppose that $a \in S_{\sharp,Y\times T_0}$. Then, for every $f \in \mathcal{V}'$, the solutions u_ϵ to (1.1) satisfy*

$$u_\epsilon \rightarrow u \text{ weakly in } \mathcal{W}_0,$$

$$a_\epsilon(x,t,Du_\epsilon) \rightarrow b(Du) \text{ weakly in } \mathcal{U}',$$

where u is the unique solution to the following parabolic problem:

$$\begin{cases} u' - \operatorname{div}(b(Du)) = f \text{ in } \Omega\times]0,T[\\ u \in \mathcal{W}_0. \end{cases} \tag{2.1}$$

Moreover, for a fixed vector $\xi \in \mathbf{R}^{\mathrm{N}}$:

$$b(\xi) = \int_{\mathcal{T}_0}\int_Y a(y,\tau,Dv(y,\tau)+\xi)\,dyd\tau, \tag{2.2}$$

where v depends on ξ and μ. For $0 < \mu < 2$, $v = v(y,\tau)$ is the unique solution to the parameter-dependent elliptic problem:

$$\begin{cases} -\operatorname{div}(a(y,\tau,Dv(y,\tau)+\xi)) = 0, \\ v(\cdot,\tau) \in \mathrm{V}_{\sharp,Y},\ \tau \geq 0. \end{cases} \tag{2.3}$$

For $\mu = 2$, $v = v(y,\tau)$ is the unique solution, to the parabolic problem:

$$\begin{cases} v' - \operatorname{div}(a(y,\tau,Dv(y,\tau)+\xi)) = 0, \\ v \in \mathcal{V}_{\sharp,Y\times T_0}. \end{cases} \tag{2.4}$$

For $\mu > 2$, finally $v = v(y)$ is the unique solution to the elliptic problem:

$$\begin{cases} -\operatorname{div}(\tilde{a}(y,Dv(y)+\xi)) = 0, \\ v \in \mathrm{V}_{\sharp,Y}, \end{cases} \tag{2.5}$$

where

$$\tilde{a}(y,\xi) = \int_{\mathcal{T}_0} a(y,\tau,\xi)d\tau. \tag{2.6}$$

Proof. We refer to [8].

Remark 1. By the estimates (4.19) and (4.21) in the proof of Theorem 3.1 in [9] it follows that the homogenized map b satisfies the estimates

$$|b(\xi_1) - b(\xi_2)| \leq C(1 + |\xi_1| + |\xi_2|)^{p-1-\gamma}|\xi_1 - \xi_2|^\gamma$$
$$(b(\xi_1) - b(\xi_2), \xi_1 - \xi_2) \geq c_2|\xi_1 - \xi_2|^p$$

for every ξ_1, $\xi_2 \in \mathbf{R}^{\mathrm{N}}$ where $\gamma = \alpha/(p - \alpha)$.

We close this section by stating some different Meyers type estimates which will be needed in the proof of the main corrector result, Theorem 3.1.

Proposition 2.3. *Suppose that $a \in S_{\sharp, Y\times T_0}$. Let u be the solution to the problem*

$$\begin{cases} -\operatorname{div}(\tilde{a}(x, Du) = 0, \\ u \in W^{1,p}(\Omega), \end{cases}$$

Then there exists a constant $\eta > 0$ such that $u \in W^{1,p+\eta}(\tilde{\Omega})$ for every open set $\tilde{\Omega} \subset\subset \Omega$. Moreover

$$\|u\|_{W^{1,p+\eta}(\tilde{\Omega})} \leq C\|u\|_{W^{1,p}(\Omega)}.$$

Proof. We refer to Theorem 1 in [7].

Remark 2. Considered as a function constant in $t \geq 0$, the function u above also satisfies the estimate

$$\|u\|_{L^{p+\eta}(0,T;W^{1,p+\eta}(\tilde{\Omega}))} \leq C\|u\|_{L^p(0,T;W^{1,p}(\Omega))}.$$

Proposition 2.4. *Suppose that $a \in S_{\sharp, Y\times T_0}$ and in addition satisfies*

$$|a(x, t, \xi) - a(x, s, \xi)| \leq \omega(t - s)(1 + |\xi|^{p-1}) \tag{2.7}$$

for all $t, s \in]0, T[$, all $\xi \in \mathbf{R}^{\mathrm{N}}$ and a.e. $x \in \Omega$, where ω is the modulus of continuity. Let $u(\cdot, t)$, $t \in]0, T[$, be the solution to the parameter dependent elliptic problem

$$\begin{cases} -\operatorname{div}(a(x, t, Du)) = 0, \\ u(\cdot, t) \in W^{1,p}(\Omega), \end{cases}$$

Then there exists a constant $\eta > 0$ such that, for every $t \in]0, T[$, $u(\cdot, t) \in W^{1,p+\eta}(\tilde{\Omega})$ for every open set $\tilde{\Omega} \subset\subset \Omega$. Moreover

$$\|u(\cdot, t)\|_{W^{1,p+\eta}(\tilde{\Omega})} \leq C\|u(\cdot, t)\|_{W^{1,p}(\Omega)}. \tag{2.8}$$

Further, let $\delta \subset\subset]0, T[$. The gradient Du of the function u above also satisfies the estimate

$$\|Du\|_{L^{p+\eta}(\delta;L^{p+\eta}(\tilde{\Omega};\mathbf{R}^{\mathrm{N}}))} \leq C\|Du\|_{L^p(0,T;L^p(\Omega;\mathbf{R}^{\mathrm{N}}))}. \tag{2.9}$$

Proof. The estimate (2.8) is a consequence of Theorem 1 in [7], if we take (2.7) into account. By using the coercivity of a and the Hölder inequality we get

$$\begin{aligned}
&\int_\delta \|Du(\cdot,t+\epsilon)-Du(\cdot,t)\|^p_{L^p(\Omega)}\,dt \\
&\quad\le c_2\int_\delta\int_\Omega (a(x,t+\epsilon,Du(x,t+\epsilon))-a(x,t,Du(x,t)),Du(x,t+\epsilon)-Du(x,t))\,dxdt \\
&\quad\le c_2(\int_\delta \|a(\cdot,t+\epsilon,Du(\cdot,t+\epsilon))-a(\cdot,t,Du(\cdot,t))\|^{p'}_{L^{p'}(\Omega)}\,dt)^{1/p'} \\
&\qquad\quad\times(\int_\delta \|Du(\cdot,t+\epsilon)-Du(\cdot,t)\|^p_{L^p(\Omega)}\,dt)^{1/p}.
\end{aligned}$$

By (2.7) we obtain, using the Minkowski inequality and the boundedness of Ω,

$$\begin{aligned}
&(\int_\delta \|Du(\cdot,t+\epsilon)-Du(\cdot,t)\|^p_{L^p(\Omega)}\,dt)^{p'} \\
&\qquad\le c_2(\int_\delta \|\omega(\epsilon)(1+|\max\{Du(\cdot,t+\epsilon),Du(\cdot,t)\}|^{p-1}\|^{p'}_{L^{p'}(\Omega)}\,dt)^{1/p'} \\
&\qquad\le c_2(\int_\delta |\omega(\epsilon)|((m(\Omega))^{p'}+\|\max\{Du(\cdot,t+\epsilon),Du(\cdot,t)\}\|^p_{L^p(\Omega)})\,dt)^{1/p'}.
\end{aligned}$$

For ϵ small enough $\omega(\epsilon)\le 1$ and (2.8) implies that

$$\sup_{t\in\delta}\|\max\{Du(\cdot,t+\epsilon),Du(\cdot,t)\}\|^p_{L^p(\Omega)}\le C,$$

where C is independent of ϵ. Therefore

$$\int_\delta \|Du(\cdot,t+\epsilon)-Du(\cdot,t)\|^p_{L^p(\Omega)}\,dt\le C\int_\delta|\omega(\epsilon)|\,dt,$$

which tends to zero as $\epsilon\to 0$, by the dominated convergence theorem. The estimate (2.9) now readily follows by the continuity of Du with respect to t. ■

Proposition 2.5. *Suppose that $a\in S_{\sharp,Y\times T_0}$. Let $u\in L^p(0,T;W^{1,p}(\Omega))\cap L^\infty(0,T;L^2(\Omega))$ be the solution to the problem*

$$u'-\operatorname{div}(a(x,t,Du)=0.$$

Let $\tilde\Omega$ be defined as above and let $\delta\subset\subset]0,T[$. The gradient Du of the function u above also satisfies the estimate

$$\|Du\|_{L^q(\delta;L^q(\tilde\Omega;\mathbf{R}^N))}\le C\|Du\|_{L^p(0,T;L^p(\Omega;\mathbf{R}^N))}.$$

for any $q\in]1,\infty[$.

Proof. We refer to Lemma 2.2 and Remark 7.4 of [5].

3 The main result

In this section we state the main corrector result which we indicated in the previous sections. We start out by defining a sequence (M_ϵ) of approximations of the identity map on $\mathcal{U}$. For $i \in \mathbf{Z}^{\mathrm{N}}$ and $j \in \mathbf{Z}$ we consider the translated images $Y_\epsilon^i = \epsilon(i+Y)$ and $T_{0,\epsilon}^j = \epsilon^\mu(j+T_0)$. Take $\varphi \in \mathcal{U}$. We define the function

$$M_\epsilon\varphi : \mathbf{R}^{\mathrm{N}} \times \mathbf{R} \to \mathbf{R}^{\mathrm{N}}$$

by

$$(M_\epsilon\varphi)(x,t) = \sum_{i\in I_\epsilon}\sum_{j\in J_\epsilon} \chi_{Y_\epsilon^i}(x)\chi_{T_{0,\epsilon}^j}(t)\frac{1}{m(Y_\epsilon^i \times T_{0,\epsilon}^j)}\int_{T_{0,\epsilon}^j}\int_{Y_\epsilon^i}\varphi(y,\tau)\,dyd\tau, \tag{3.1}$$

where

$$I_\epsilon = \{i \in \mathbf{Z}^{\mathbf{N}} : Y_\epsilon^i \in \Omega\} \text{ and } J_\epsilon = \{j \in \mathbf{Z} : T_{0,\epsilon}^j \in]0,\mathrm{T}[\}$$

and χ_A denotes the characteristic function of the measurable set A. It is well-known that

$$M_\epsilon\varphi \to \varphi \;\text{ strongly in }\; \mathcal{U}. \tag{3.2}$$

By the Jensen's inequality we also have

$$\|M_\epsilon\varphi\|_{\mathcal{U}} \leq \|\varphi\|_{\mathcal{U}} \tag{3.3}$$

for all $\varphi \in \mathcal{U}$.

Let us also define the $Y \times T_0$-periodic function

$$p : \mathbf{R}^{\mathrm{N}} \times \mathbf{R} \times \mathbf{R}^{\mathrm{N}} \to \mathbf{R}^{\mathrm{N}},$$

which depends on μ, by

$$p(x,t,\xi) = \xi + Dv(x,t), \tag{3.4}$$

where v is the solution to the auxiliary local problem (2.3), (2.4) or (2.5) for $0 < \mu < 2$, $\mu = 2$ and $\mu > 2$, respectively. It follows that the function

$$p_\epsilon : \mathbf{R}^{\mathrm{N}} \times \mathbf{R} \times \mathbf{R}^{\mathrm{N}} \to \mathbf{R}^{\mathrm{N}}$$

defined by

$$p_\epsilon(x,t,\xi) = \xi + Dv(\frac{x}{\epsilon},\frac{t}{\epsilon^\mu}), \tag{3.5}$$

is ϵY-periodic in x and $\epsilon^\mu T_0$-periodic in t. This means that

$$\int_{T_0}\int_Y p(x,t,\xi)\,dxdt = \xi$$

and that

$$p_\epsilon(\cdot,\cdot,\xi) \to \xi \;\text{ weakly in }\; \mathcal{U}. \tag{3.6}$$

Thus, the homogenized map b can be expressed as

$$b(\xi) = \int_{T_0} \int_Y a(x, t, p(x, t, \xi))\, dx dt. \tag{3.7}$$

Moreover, we have

$$\begin{aligned}\int_{T_0} \int_Y &(a(x, t, p(x, t, \xi)), p(x, t, \xi))\, dx dt \\ &= \int_{T_0} \int_Y (a(x, t, p(x, t, \xi)), \xi)\, dx dt = (b(\xi), \xi).\end{aligned} \tag{3.8}$$

The following correctors result is the main result of this paper:

Theorem 3.1. *Suppose that $a \in S_{\sharp, Y \times T_0}$. For the case $0 < \mu < 2$ we also suppose that a satisfies (2.7). Moreover, suppose that $f \in \mathcal{V}'$ and let u_ϵ be the solutions to* (1.1) *and let u be the solution to* (1.2). *Then, we have*

$$Du_\epsilon = p_\epsilon(\cdot, \cdot, M_\epsilon Du) + r_\epsilon, \tag{3.9}$$

where p_ϵ is defined by (3.5) *and where*

$$r_\epsilon \to 0, \quad \textit{strongly in } \mathcal{U}.$$

Remark 3. Recall that p_ϵ is entirely different for the three cases $0 < \mu < 2$, $\mu = 2$ and $\mu > 2$, respectively.

4 Some estimates for the family of correctors

In this section we present some estimates for the family (p_ϵ) of correctors. To a large extent the proofs will follow by minor modifications of the proofs of similar lemmas by Dal Maso and Defranceschi in [4]. Therefore we refer to their paper for complete details and present here only proofs of parts which require more modifications.

Lemma 4.1. *For any vector $\xi \in \mathbf{R}^{\mathrm{N}}$ we have*

$$\|p_\epsilon(\cdot, \cdot, \xi)\|^p_{L^p(T_{0,\epsilon}; L^p(Y_\epsilon; \mathbf{R}^{\mathrm{N}}))} \le C(1 + |\xi|^p) m(Y_\epsilon \times T_{0,\epsilon}), \tag{4.1}$$

where the constant C depends only on N, *p, c_0, c_1 and c_2.*

Lemma 4.2. *There exist $\eta > 0$ and $C > 0$, which depends only on* N, *p, c_0, c_1 and c_2, such that*

$$\|p_\epsilon(\cdot, \cdot, \xi)\|^{p+\eta}_{L^{p+\eta}(T_{0,\epsilon}; L^{p+\eta}(Y_\epsilon; \mathbf{R}^{\mathrm{N}}))} \le C(1 + |\xi|^{p+\eta}) m(Y_\epsilon \times T_{0,\epsilon}), \tag{4.2}$$

for every $\xi \in \mathbf{R}^{\mathrm{N}}$.

Proof. By referring to the Meyers estimates in Propositions 2.3, 2.4 and 2.5 the proof is analogous as the proof of Corollary 3.3 in [4]. ■

Lemma 4.3. *For every ξ_1, ξ_2 in $\mathbf{R}^{\mathrm{N}}$ we have*

$$\|p_\epsilon(\cdot,\cdot,\xi_1) - p_\epsilon(\cdot,\cdot,\xi_2)\|^p_{L^p(T_{0,\epsilon};L^p(Y_\epsilon;\mathbf{R}^{\mathrm{N}}))}$$
$$\leq C(1+|\xi_1|^p+|\xi_2|^p)^{(p-1-\alpha)/(p-\alpha)}|\xi_1-\xi_2|^{p/(p-\alpha)}m(Y_\epsilon \times T_{0,\epsilon}), \tag{4.3}$$

where the constant C depends only on N, p, α, c_0, c_1 *and* c_2.

Lemma 4.4. *Let $\varphi \in \mathcal{U}$ and consider a simple function Ψ given by*

$$\Psi(x,t) = \sum_{k=1}^m c_k \chi_{\Omega_k}(x)\chi_{\delta_k}(t),$$

with $c_k \in \mathbf{R}^{\mathrm{N}} \setminus \{0\}$, $\Omega_k \subset\subset \Omega$, $\delta_k \subset\subset]0,\mathrm{T}[$, $m(\partial\Omega_k) = m(\partial\delta_k) = 0$ and $(\Omega_k \cap \Omega_l) \times (\delta_k \cap \delta_l) = \phi$ for $k \neq l$. Then

$$\limsup_{\epsilon\to 0} \|p_\epsilon(\cdot,\cdot,M_\epsilon\varphi) - p_\epsilon(\cdot,\cdot,\Psi)\|_{\mathcal{U}}$$
$$\leq C(m(\Omega\times]0,\mathrm{T}[) + \|\varphi\|_{\mathcal{U}} + \|\Psi\|_{\mathcal{U}})^{(p-1-\alpha)/(p-\alpha)}\|\varphi-\Psi\|_{\mathcal{U}}^{1/(p-\alpha)}, \tag{4.4}$$

where C depends only on N, p, α, c_0, c_1 *and* c_2.

Proof. Put $\Omega_0 = \Omega \setminus \cup_{k=1}^m \Omega_k$, $\delta_0 =]0,\mathrm{T}[\setminus \cup_{k=1}^m \delta_k$ and $c_0 = 0$. Then we have

$$\Psi(x,t) = \sum_{k=0}^m c_k \chi_{\Omega_k}(x)\chi_{\delta_k}(t).$$

For every $\epsilon > 0$ we denote by $\Omega_\epsilon \times \delta_\epsilon$ the union of all closed cubes $\overline{Y}_\epsilon^i \times \overline{T}_{0,\epsilon}^j$ such that $Y_\epsilon^i \subset \Omega$ and $T_{0,\epsilon}^j \subset]0,\mathrm{T}[$. For $k = 0, 1, \ldots, m$ we define the sets

$$I_\epsilon^k = \{i \in I_\epsilon : Y_\epsilon^i \subset \Omega_k\}, \quad J_\epsilon^k = \{j \in J_\epsilon : T_{0,\epsilon}^i \subset \delta_k\},$$

and

$$\tilde{I}_\epsilon^k = \{i \in I_\epsilon : Y_\epsilon^i \cap \Omega_k \neq \phi,\ Y_\epsilon^i \setminus \Omega_k \neq \phi\},$$
$$\tilde{J}_\epsilon^k = \{j \in J_\epsilon : T_{0,\epsilon}^i \cap \delta_k \neq \phi,\ T_{0,\epsilon}^i \setminus \delta_k \neq \phi\}.$$

Further, we define $E_\epsilon^{i,j,k}$ as the union of all closed cubes $\overline{Y}_\epsilon^i \times \overline{T}_{0,\epsilon}^j$ with $i \in I_\epsilon^k$ and $j \in J_\epsilon^k$, and we define $\tilde{E}_\epsilon^{i,j,k}$ as the union of all closed cubes $\overline{Y}_\epsilon^i \times \overline{T}_{0,\epsilon}^j$ with $i \in \tilde{I}_\epsilon^k$ and $j \in \tilde{J}_\epsilon^k$. If we choose ϵ small enough, then, for $k \neq 0$, $\Omega_k \times \delta_k \subseteq \Omega_\epsilon \times \delta_\epsilon$ according to (3.1). Thus, the definition of Ψ yields

$$\|p_\epsilon(\cdot,\cdot,M_\epsilon\varphi) - p_\epsilon(\cdot,\cdot,\Psi)\|^p_{\mathcal{U}} = \int_{\delta_\epsilon}\int_{\Omega_\epsilon} |p_\epsilon(x,t,M_\epsilon\varphi) - p_\epsilon(x,t,\Psi)|^p\,dxdt$$
$$\leq \sum_{k=0}^m \int_{E_\epsilon^{i,j,k}} |p_\epsilon(x,t,M_\epsilon\varphi) - p_\epsilon(x,t,c_k)|^p\,dxdt$$
$$+ \sum_{k=0}^m \int_{\tilde{E}_\epsilon^{i,j,k}} |p_\epsilon(x,t,M_\epsilon\varphi) - p_\epsilon(x,t,c_k)|^p\,dxdt.$$

Let us put

$$\theta_\epsilon^{i,j} = \frac{1}{m(Y_\epsilon^i \times T_{0,\epsilon}^j)} \int_{T_{0,\epsilon}^j} \int_{Y_\epsilon^i} \varphi(x,t)\, dxdt.$$

A repeated application of the Hölder's and the Jensen's inequalities yields, according to Lemma 4.3,

$$\begin{aligned}
&\|p_\epsilon(\cdot,\cdot,M_\epsilon\varphi) - p_\epsilon(\cdot,\cdot,\Psi)\|_{\mathcal{U}}^p \\
&\quad\le C\sum_{k=0}^m \left(\sum_{j\in J_\epsilon^k} \sum_{i\in I_\epsilon^k} (1+|\theta_\epsilon^{i,j}|^p + |c_k|^p)^{(p-1-\alpha)/(p-\alpha)} |\theta_\epsilon^{i,j} - c_k|^{p/(p-\alpha)} m(Y_\epsilon^i \times T_{0,\epsilon}^j) \right) \\
&\quad+ C\sum_{k=0}^m \left(\sum_{j\in \tilde{J}_\epsilon^k} \sum_{i\in \tilde{I}_\epsilon^k} (1+|\theta_\epsilon^{i,j}|^p + |c_k|^p)^{(p-1-\alpha)/(p-\alpha)} |\theta_\epsilon^{i,j} - c_k|^{p/(p-\alpha)} m(Y_\epsilon^i \times T_{0,\epsilon}^j) \right) \\
&\quad\le C\left(\sum_{k=0}^m \left(m(E_\epsilon^{i,j,k}) + \int_{E_\epsilon^{i,j,k}} |\varphi|^p\, dxdt + |c_k|^p m(E_\epsilon^{i,j,k}) \right) \right)^{\frac{p-1-\alpha}{p-\alpha}} \|\varphi-\Psi\|_{\mathcal{U}}^{p/(p-\alpha)} \\
&\quad+ C\sum_{k=0}^m \left(\left(m(\tilde{E}_\epsilon^{i,j,k}) + \int_{\tilde{E}_\epsilon^{i,j,k}} |\varphi|^p\, dxdt + |c_k|^p m(\tilde{E}_\epsilon^{i,j,k}) \right)^{\frac{p-1-\alpha}{p-\alpha}} \|\varphi - c_k\|_{\mathcal{U}}^{p/(p-\alpha)} \right).
\end{aligned}$$

Hence

$$\begin{aligned}
&\|p_\epsilon(\cdot,\cdot,M_\epsilon\varphi) - p_\epsilon(\cdot,\cdot,\Psi)\|_{\mathcal{U}}^p \\
&\quad\le C(m(\Omega\times]0,\mathrm{T}[) + \|\varphi\|_{\mathcal{U}}^p + \|\Psi\|_{\mathcal{U}}^p)^{(p-1-\alpha)/(p-\alpha)} \|\varphi-\Psi\|_{\mathcal{U}}^{p/(p-\alpha)} \\
&\quad+ C\sum_{k=0}^m \left(\left(m(\tilde{E}_\epsilon^{i,j,k}) + \int_{\tilde{E}_\epsilon^{i,j,k}} |\varphi|^p\, dxdt + |c_k|^p m(\tilde{E}_\epsilon^{i,j,k}) \right)^{\frac{p-1-\alpha}{p-\alpha}} \|\varphi - c_k\|_{\mathcal{U}}^{p/(p-\alpha)} \right).
\end{aligned} \tag{4.5}$$

Now recall that $m(\partial\Omega_k) = m(\partial\delta_k) = 0$ for $k \neq 0$. Thus, $m(\tilde{E}_\epsilon^{i,j,k}) \to 0$ as $\epsilon \to 0$ for every $k = 0, 1, \ldots, m$ and the lemma is proved. ∎

5 Proof of the main corrector result

In this section we give the proof of the main corrector result, Theorem 3.1, stated in Section 3. Our proof will follow the lines of the proof of the corrector result for the corresponding elliptic problem, earlier proved by Dal Maso and Defranceschi in [4]. We start out by proving an estimate on $p_\epsilon(\cdot,\cdot,M_\epsilon Du)$ uniformly with respect to ϵ.

Lemma 5.1. *Let p_ϵ be defined as in* (3.5)*. Then,*

$$\|p_\epsilon(\cdot,\cdot,M_\epsilon Du)\|_{\mathcal{U}}^p \le C, \tag{5.1}$$

where the positive constant C is independent of ϵ.

Proof. Let us define

$$\theta_\epsilon^{i,j} = \frac{1}{m(Y_\epsilon^i \times T_{0,\epsilon}^j)} \int_{T_{0,\epsilon}^j} \int_{Y_\epsilon^i} Du(x,t)\, dxdt.$$

and

$$\tilde{I}_\epsilon = \{i \in \mathbf{Z}^{\mathrm{N}} : Y_\epsilon^i \cap \Omega \neq \phi,\ Y_\epsilon^i \setminus \Omega \neq \phi\},$$
$$\tilde{J}_\epsilon = \{j \in \mathbf{Z} : T_{0,\epsilon}^i \cap]0,\mathrm{T}[\neq \phi,\ T_{0,\epsilon}^i \setminus]0,\mathrm{T}[\neq \phi\}.$$

We apply Lemma 4.1, Lemma 4.2 and the inequality (3.3) to obtain

$$\begin{aligned}
&\|p_\epsilon(\cdot,\cdot,M_\epsilon Du)\|_{\mathcal{U}}^p \\
&\quad = \sum_{j\in J_\epsilon}\sum_{i\in I_\epsilon} \int_{T_{0,\epsilon}^j}\int_{Y_\epsilon^i} |p_\epsilon(x,t,\theta_\epsilon^{i,j})|^p\, dxdt + \int_{]0,\mathrm{T}[\setminus\delta_\epsilon}\int_{\Omega\setminus\Omega_\epsilon} |p_\epsilon(x,t,0)|^p\, dxdt \\
&\quad \le \sum_{j\in J_\epsilon}\sum_{i\in I_\epsilon} C(1+|\theta_\epsilon^{i,j}|^p) m(Y_\epsilon^i \times T_{0,\epsilon}^j) \\
&\qquad + \left(\sum_{j\in\tilde{J}_\epsilon}\sum_{i\in\tilde{I}_\epsilon} \|p_\epsilon(\cdot,\cdot,0)\|^{p+\eta}_{L^{p+\eta}(T_{0,\epsilon}^j;L^{p+\eta}(Y_\epsilon^i;\mathbf{R}^{\mathrm{N}}))}\right)^{p/(p+\eta)} \\
&\qquad \times m((\Omega\setminus\Omega_\epsilon)\times(]0,\mathrm{T}[\setminus\delta_\epsilon))^{\eta/(p+\eta)} \\
&\quad \le Cm(\Omega\times]0,\mathrm{T}[) + C\|M_\epsilon Du\|_{\mathcal{U}}^p + C\left(\sum_{j\in\tilde{J}_\epsilon}\sum_{i\in\tilde{I}_\epsilon} m(Y_\epsilon^i\times T_{0,\epsilon}^j)\right)^{p/(p+\eta)} \\
&\qquad \times m((\Omega\setminus\Omega_\epsilon)\times(]0,\mathrm{T}[\setminus\delta_\epsilon))^{\eta/(p+\eta)} \\
&\quad \le Cm(\Omega\times]0,\mathrm{T}[) + C\|Du\|_{\mathcal{U}}^p + C\left(\sum_{j\in\tilde{J}_\epsilon}\sum_{i\in\tilde{I}_\epsilon} m(Y_\epsilon^i\times T_{0,\epsilon}^j)\right)^{p/(p+\eta)} \\
&\qquad \times m((\Omega\setminus\Omega_\epsilon)\times(]0,\mathrm{T}[\setminus\delta_\epsilon))^{\eta/(p+\eta)}. \qquad (5.2)
\end{aligned}$$

Now $\sum_{j\in\tilde{J}_\epsilon}\sum_{i\in\tilde{I}_\epsilon} m(Y_\epsilon^i \times T_{0,\epsilon}^j)$ approaches $m(\partial\Omega\times\partial]0,\mathrm{T}[)$ and $m((\Omega\setminus\Omega_\epsilon)\times(]0,\mathrm{T}[\setminus\delta_\epsilon))$ tends to zero as $\epsilon\to 0$. Thus, (5.1) follows by (5.2) and Lemma 5.1 is proved. ■

Proof of Theorem 3.1. By the strict monotonicity assumption it follows that

$$\begin{aligned}
&\|p_\epsilon(\cdot,\cdot,M_\epsilon Du) - Du_\epsilon\|_{\mathcal{U}} \\
&\quad \le C\left(\int_0^T\int_\Omega (a_\epsilon(x,t,p_\epsilon(x,t,M_\epsilon Du)) - a_\epsilon(x,t,Du_\epsilon), p_\epsilon(x,t,M_\epsilon Du) - Du_\epsilon)\, dxdt\right)^{1/p}. \qquad (5.3)
\end{aligned}$$

Consequently, Theorem 3.1 is proved if we can prove that

$$\int_0^T\int_\Omega (a_\epsilon(x,t,p_\epsilon(x,t,M_\epsilon Du)) - a_\epsilon(x,t,Du_\epsilon), p_\epsilon(x,t,M_\epsilon Du) - Du_\epsilon)\, dxdt \to 0 \quad (5.4)$$

as $\epsilon \to 0$. The proof of (5.4) will be splitted up into four steps.

Step 1. We start by showing that

$$\int_0^T \int_\Omega (a_\epsilon(x,t,p_\epsilon(x,t,M_\epsilon Du)), p_\epsilon(x,t,M_\epsilon Du))\,dxdt \to \int_0^T \int_\Omega (b(Du), Du)\,dxdt. \tag{5.5}$$

Let us write

$$\begin{aligned}
&\int_0^T \int_\Omega (a_\epsilon(x,t,p_\epsilon(x,t,M_\epsilon Du)), p_\epsilon(x,t,M_\epsilon Du))\,dxdt \\
&\quad = \sum_{j\in J_\epsilon}\sum_{i\in I_\epsilon}\int_{T^j_{0,\epsilon}}\int_{Y^i_\epsilon}(a(\frac{x}{\epsilon},\frac{t}{\epsilon^\mu},p(\frac{x}{\epsilon},\frac{t}{\epsilon^\mu},\theta^{i,j}_\epsilon)),p(\frac{x}{\epsilon},\frac{t}{\epsilon^\mu},\theta^{i,j}_\epsilon))\,dxdt \\
&\qquad + \int_{]0,\mathrm{T}[\setminus\delta_\epsilon}\int_{\Omega\setminus\Omega_\epsilon}(a_\epsilon(x,t,p_\epsilon(x,t,0)),p_\epsilon(x,t,0))\,dxdt \\
&\quad = \epsilon^{\mathrm{N}+1}\sum_{j\in J_\epsilon}\sum_{i\in I_\epsilon}\int_{T_0}\int_{Y}(a(y,\tau,p(y,\tau,\theta^{i,j}_\epsilon)),p(y,\tau,\theta^{i,j}_\epsilon))\,dyd\tau \\
&\qquad + \int_{]0,\mathrm{T}[\setminus\delta_\epsilon}\int_{\Omega\setminus\Omega_\epsilon}(a_\epsilon(x,t,p_\epsilon(x,t,0)),p_\epsilon(x,t,0))\,dxdt \\
&\quad = \sum_{j\in J_\epsilon}\sum_{i\in I_\epsilon}\int_0^T\int_\Omega \chi_{Y^i_\epsilon}(y)\chi_{T^j_{0,\epsilon}}(t)(b(\theta^{i,j}_\epsilon),\theta^{i,j}_\epsilon)\,dyd\tau \\
&\qquad + \int_{]0,\mathrm{T}[\setminus\delta_\epsilon}\int_{\Omega\setminus\Omega_\epsilon}(a_\epsilon(x,t,p_\epsilon(x,t,0)),p_\epsilon(x,t,0))\,dxdt,
\end{aligned}\tag{5.6}$$

where the last equality follows from (3.8). According to Remark 1 the map $\varphi \to b(\varphi)$ is continuous from $\mathcal{U}$ into $\mathcal{U}'$ and an application of (3.2), using this fact, yields

$$b(M_\epsilon Du) \to b(Du) \;\text{ strongly in }\; \mathcal{U}'. \tag{5.7}$$

and, thus,

$$\begin{aligned}
&\sum_{j\in J_\epsilon}\sum_{i\in I_\epsilon}\int_0^T\int_\Omega \chi_{Y^i_\epsilon}(y)\chi_{T^j_{0,\epsilon}}(t)(b(\theta^{i,j}_\epsilon),\theta^{i,j}_\epsilon)\,dyd\tau \\
&\quad = \int_0^T\int_\Omega (b(M_\epsilon Du, M_\epsilon Du)\,dyd\tau \to \int_0^T\int_\Omega (b(Du,Du)\,dyd\tau.
\end{aligned}\tag{5.8}$$

By the uniform continuity assumption we have

$$\begin{aligned}
&|\int_{]0,\mathrm{T}[\setminus\delta_\epsilon}\int_{\Omega\setminus\Omega_\epsilon}(a_\epsilon(x,t,p_\epsilon(x,t,0)),p_\epsilon(x,t,0))\,dxdt| \\
&\quad \le C\int_{]0,\mathrm{T}[\setminus\delta_\epsilon}\int_{\Omega\setminus\Omega_\epsilon}(1+|p_\epsilon(x,t,0)|)^p\,dxdt \\
&\qquad + |\int_{]0,\mathrm{T}[\setminus\delta_\epsilon}\int_{\Omega\setminus\Omega_\epsilon}(a_\epsilon(x,t,0),p_\epsilon(x,t,0))\,dxdt|
\end{aligned}$$

$$\leq Cm((\Omega \setminus \Omega_\epsilon) \times (]0,\mathrm{T}[\setminus \delta_\epsilon)) + C \int_{]0,\mathrm{T}[\setminus \delta_\epsilon} \int_{\Omega \setminus \Omega_\epsilon} |p_\epsilon(x,t,0)|^p \, dxdt$$

$$+ C \left(\sum_{j \in J_\epsilon} \sum_{i \in I_\epsilon} m(Y^i_\epsilon \times T^j_{0,\epsilon}) \right)^{1/p'} \left(\int_{]0,\mathrm{T}[\setminus \delta_\epsilon} \int_{\Omega \setminus \Omega_\epsilon} |p_\epsilon(x,t,0)|^p \, dxdt \right)^{1/p}.$$

By arguing as in Lemma 5.1 we conclude that

$$| \int_{]0,\mathrm{T}[\setminus \delta_\epsilon} \int_{\Omega \setminus \Omega_\epsilon} (a_\epsilon(x,t,p_\epsilon(x,t,0)), p_\epsilon(x,t,0)) \, dxdt| \to 0.$$

Thus, by taking (5.6) and (5.8) into account we have shown (5.5).

Step 2. We proceed by showing that

$$\int_0^T \int_\Omega (a_\epsilon(x,t,p_\epsilon(x,t,M_\epsilon Du)), Du_\epsilon) \, dxdt \to \int_0^T \int_\Omega (b(Du), Du) \, dxdt. \tag{5.9}$$

Let $\rho > 0$ be arbitrary. For $Du \in \mathcal{U}$ there exists a simple function

$$\Psi = \sum_{k=1}^m c_k \chi_{\Omega_k} \chi_{\delta_k},$$

which satisfies the assumptions in Lemma 4.4, such that

$$\|Du - \Psi\|_{\mathcal{U}} \leq \rho. \tag{5.10}$$

We write

$$\begin{aligned}
&\int_0^T \int_\Omega (a_\epsilon(x,t,p_\epsilon(x,t,M_\epsilon Du)), Du_\epsilon) \, dxdt \\
&\quad = \int_0^T \int_\Omega (a_\epsilon(x,t,p_\epsilon(x,t,\Psi)), Du_\epsilon) \, dxdt \\
&\quad\quad + \int_0^T \int_\Omega (a_\epsilon(x,t,p_\epsilon(x,t,M_\epsilon Du)) - (a_\epsilon(x,t,p_\epsilon(x,t,\Psi)), Du_\epsilon) \, dxdt.
\end{aligned} \tag{5.11}$$

It follows, for the first integral on the right hand side, that

$$\begin{aligned}
&\int_0^T \int_\Omega (a_\epsilon(x,t,p_\epsilon(x,t,\Psi)), Du_\epsilon) \, dxdt \\
&\quad = \sum_{k=0}^m \int_{\delta_k} \int_{\Omega_k} (a_\epsilon(x,t,p_\epsilon(x,t,c_k)), Du_\epsilon) \, dxdt,
\end{aligned} \tag{5.12}$$

where $c_0 = 0$ and where Ω_0 and δ_0 are defined as in the previous section. By Lemma 4.2, the functions $p_\epsilon(\cdot,\cdot,c_k))$ are bounded in $L^{p+\eta}(0,T;L^{p+\eta}(\Omega;\mathbf{R}^\mathrm{N}))$. By the structure conditions this implies that $a_\epsilon(\cdot,\cdot,p_\epsilon(\cdot,\cdot,c_k)$ is uniformly bounded in $L^s(0,T;L^s(\Omega;\mathbf{R}^\mathrm{N}))$ for some $s > p'$. From Proposition 2.2 it further follows that the sequence (Du_ϵ) is bounded in $\mathcal{U}$. Therefore there exists a number $\sigma > 1$ such that

$$\|(a_\epsilon(\cdot,\cdot,p_\epsilon(\cdot,\cdot,c_k)), Du_\epsilon)\|_{L^\sigma(]0,T[\times\Omega)} \leq C$$

uniformly with respect to ϵ. Hence, up to a subsequence,

$$(a_\epsilon(\cdot,\cdot,p_\epsilon(\cdot,\cdot,c_k)), Du_\epsilon) \to g_k \text{ weakly in } L^\sigma(]0,T[\times\Omega),$$

as $\epsilon \to 0$. By proposition 2.2 we know that

$$a_\epsilon(\cdot,\cdot,p_\epsilon(\cdot,\cdot,c_k)) \to b(c_k) \text{ weakly in } \mathcal{U}'.$$

This enables us to use the compensated compactness result Theorem 2.1 in [9] and conclude that

$$(a_\epsilon(\cdot,\cdot,p_\epsilon(\cdot,\cdot,c_k)), Du_\epsilon) \to (b(c_k), Du)$$

in the sense of distributions. Consequently $g_k = (b(c_k), Du)$ and

$$\sum_{k=0}^{m}\int_{\delta_k}\int_{\Omega_k}(a_\epsilon(x,t,p_\epsilon(x,t,c_k)), Du_\epsilon)\,dxdt \to \sum_{k=0}^{m}\int_{\delta_k}\int_{\Omega_k}(b(c_k), Du)\,dxdt.$$

By using (5.12) this gives

$$\int_0^T\int_\Omega(a_\epsilon(x,t,p_\epsilon(x,t,\Psi)), Du_\epsilon)\,dxdt \to \int_0^T\int_\Omega(b(\Psi), Du)\,dxdt. \tag{5.13}$$

For the second integral on the right hand side of (5.11) we observe that the growth condition on a_ϵ together with the Hölder inequality gives

$$\begin{aligned}
&|\int_0^T\int_\Omega(a_\epsilon(x,t,p_\epsilon(x,t,M_\epsilon Du)) - (a_\epsilon(x,t,p_\epsilon(x,t,\Psi))), Du_\epsilon)\,dxdt| \\
&\quad\le C\int_0^T\int_\Omega(1+|p_\epsilon(x,t,M_\epsilon Du)|^p + |p_\epsilon(x,t,\Psi)|^p)^{(p-1-\alpha)/p} \\
&\qquad\times |p_\epsilon(x,t,M_\epsilon Du) - p_\epsilon(x,t,\Psi)|^\alpha |Du_\epsilon|\,dxdt \\
&\quad\le C(\int_0^T\int_\Omega(1+|p_\epsilon(x,t,M_\epsilon Du)|^p + |p_\epsilon(x,t,\Psi)|^p)^{(p-1-\alpha)/p}) \\
&\qquad\times(\int_0^T\int_\Omega|Du_\epsilon|^p\,dxdt)^{1/p}|p_\epsilon(x,t,M_\epsilon Du) - p_\epsilon(x,t,\Psi)|^p\,dxdt)^{\alpha/p}.
\end{aligned} \tag{5.14}$$

By the Lemmas 5.1 and 4.1 the sequences $(p_\epsilon(\cdot,\cdot,M_\epsilon Du))$ and $(p_\epsilon(\cdot,\cdot,\Psi))$ are bounded in $\mathcal{U}$. Therefore, by using (5.10), the last inequality in (5.14) and Lemma 4.4 gives

$$\limsup_{\epsilon\to 0}|\int_0^T\int_\Omega(a_\epsilon(x,t,p_\epsilon(x,t,M_\epsilon Du)) - (a_\epsilon(x,t,p_\epsilon(x,t,\Psi))), Du_\epsilon)\,dxdt| \le C\rho^\gamma. \tag{5.15}$$

By taking Remark 1 into account we obtain

$$\begin{aligned}
&|\int_0^T\int_\Omega(b(Du) - b(\Psi), Du)\,dxdt| \\
&\quad\le C\int_0^T\int_\Omega(1+|Du|^p + |\psi|^p)^{(p-1-\gamma)/p}|Du-\Psi|^\gamma|Du|\,dxdt.
\end{aligned}$$

Again using the Hölder inequality, and (5.10), yields

$$
\begin{aligned}
&|\int_0^T \int_\Omega (b(Du) - b(\Psi), Du)\, dxdt| \\
&\qquad \le C(\int_0^T \int_\Omega (1 + |Du|^p + |\psi|^p)\, dxdt)^{(p-1-\gamma)/p} (\int_0^T \int_\Omega |Du|^p\, dxdt)^{1/p} \rho^\gamma \le C\rho^\gamma.
\end{aligned}
\tag{5.16}
$$

Thus (5.9) follows by the arbitrariness of ρ and Step 2 is accomplished.

Step 3 We show that

$$
\int_0^T \int_\Omega (a_\epsilon(x,t,Du_\epsilon)), p_\epsilon(x,t,M_\epsilon Du))\, dxdt \to \int_0^T \int_\Omega (b(Du), Du)\, dxdt. \tag{5.17}
$$

Let us fix $\delta > 0$ and let Ψ be defined as in Step 2. We write

$$
\begin{aligned}
&\int_0^T \int_\Omega (a_\epsilon(x,t,Du_\epsilon)), p_\epsilon(x,t,M_\epsilon Du)\, dxdt \\
&\qquad = \sum_{k=0}^m \int_{\delta_k} \int_{\Omega_k} (a_\epsilon(x,t,Du_\epsilon(x,t,c_k)), p_\epsilon(x,t,c_k))\, dxdt \\
&\qquad\quad + \int_0^T \int_\Omega (a_\epsilon(x,t,Du_\epsilon), p_\epsilon(x,t,M_\epsilon Du) - p_\epsilon(x,t,\Psi)\, dxdt.
\end{aligned}
\tag{5.18}
$$

By similar arguments as in Step 2 we conclude that

$$
\sum_{k=0}^m \int_{\delta_k} \int_{\Omega_k} (a_\epsilon(x,t,Du_\epsilon(x,t,c_k)), p_\epsilon(x,t,c_k))\, dxdt \to \int_0^T \int_\Omega (b(Du), \Psi)\, dxdt. \tag{5.19}
$$

It also follows, by the Hölder inequality, that

$$
\begin{aligned}
&|\int_0^T \int_\Omega (a_\epsilon(x,t,Du_\epsilon), p_\epsilon(x,t,M_\epsilon Du) - p_\epsilon(x,t,\Psi))\, dxdt| \\
&\qquad \le (\int_0^T \int_\Omega |a_\epsilon(x,t,Du_\epsilon)|^{p'}\, dxdt)^{1/p'} \\
&\qquad\quad \times (\int_0^T \int_\Omega |p_\epsilon(x,t,M_\epsilon Du) - p_\epsilon(x,t,\Psi)|^p\, dxdt)^{1/p}.
\end{aligned}
$$

Therefore, according to Lemma 4.4,

$$
\limsup_{\epsilon \to 0} |\int_0^T \int_\Omega (a_\epsilon(x,t,Du_\epsilon), p_\epsilon(x,t,M_\epsilon Du) - p_\epsilon(x,t,\Psi))\, dxdt| \le C\rho^{1/p-\alpha}. \tag{5.20}
$$

(5.17) now follows by an analogous argumentation as in the final lines of Step 2.

Step 4 In order to conclude the proof let us show that

$$
\int_0^T \int_\Omega (a_\epsilon(x,t,Du_\epsilon)), Du_\epsilon))\, dxdt \to \int_0^T \int_\Omega (b(Du), Du)\, dxdt. \tag{5.21}
$$

First we observe that

$$\int_0^T \int_\Omega (a_\epsilon(x,t,Du_\epsilon)), Du_\epsilon))\,dxdt = -\langle u'_\epsilon, u_\epsilon\rangle + \langle f, u_\epsilon\rangle,$$

or equivalently

$$\int_0^T \int_\Omega (a_\epsilon(x,t,Du_\epsilon)), Du_\epsilon))\,dxdt = -\frac{1}{2}(\|u_\epsilon(T)\|^2_{L^2(\Omega)} - \|u_\epsilon(0)\|^2_{L^2(\Omega)}) + \langle f, u_\epsilon\rangle.$$

Since $\mathcal{W}_0$ is continuously embedded in $C(0,T;L^2(\Omega))$ we can pass to the limit in the right hand side and, consequently,

$$\int_0^T \int_\Omega (b(Du), Du)\,dxdt = -\frac{1}{2}(\|u(T)\|^2_{L^2(\Omega)} - \|u(0)\|^2_{L^2(\Omega)}) + \langle f, u\rangle.$$

By collecting the results from the Steps 1-4 (5.21) follows and the proof is complete. ∎

Remark 4. The results of Theorem 3.1 remain valid even for non-homogeneous or even more general boundary data. This follows from Theorem 6.1 in [9]. We can also allow oscillating right hand side and initial data, c.f. Theorem 4.1 and Remark 6.1 in [9].

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